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Almost periodic Verblunsky coefficients and reproducing kernels on Riemann surfaces

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Dedicated with great pleasure to Barry Simon on the occasion of his 60th birthday

Abstract

We give an explicit parametrization of a set of almost periodic CMV matrices whose spectrum (is equal to the absolute continuous spectrum and) is a homogenous set E lying on the unit circle, for instance a Cantor set of positive Lebesgue measure. First to every operator of this set we associate a function from a certain subclass of the Schur functions. Then it is shown that such a function can be represented by reproducing kernels of appropriated Hardy spaces and, consequently, it gives rise to a CMV matrix of the set under consideration. If E is a finite system of arcs our results become basically the results of Geronimo and Johnson.

1. Introduction

A closed subset E of the unit circle $\mathbb{T}$ is called homogeneous if there is $\eta > 0$ such that

$$|(e^{i(\theta-\delta)}, e^{i(\theta+\delta)}) \cap E| \geqslant \eta\delta$$

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for all $0 < \delta < \pi$ and all $e^{i\theta} \in E$. A finite union of (necessary non-degenerate) arcs on the unit circle $\mathbb{T}$, of course, is a homogeneous set. A non-trivial example of a set with this property is a standard Cantor set of positive length (concerning this and other properties of homogeneous sets see [2,10], see also [22,18]).

For a given sequence of numbers from the unit disk $\mathbb{D}$

$$\ldots, a_{-1}, a_0, a_1, a_2, \ldots \tag{1}$$

define unitary 2×2 matrices

$$A_k = \begin{bmatrix} \overline{a_k} & \rho_k \\ \rho_k & -a_k \end{bmatrix}, \quad \rho_k = \sqrt{1 - |a_k|^2}$$

and unitary operators in $l^2(\mathbb{Z})$ given by block-diagonal matrices

$$\mathfrak{A}_0 = \begin{bmatrix} \ddots & & & \\ & A_{-2} & & \\ & & A_0 & \\ & & & \ddots \end{bmatrix}, \quad \mathfrak{A}_1 = S \begin{bmatrix} \ddots & & & \\ & A_{-1} & & \\ & & A_1 & \\ & & & \ddots \end{bmatrix} S^{-1},$$

where $S|k\rangle = |k+1\rangle$. The CMV matrix $\mathfrak{A}$, generated by sequence (1), is the product

$$\mathfrak{A} = \mathfrak{A}(\{a_k\}) := \mathfrak{A}_0 \mathfrak{A}_1. \tag{2}$$

CMV matrices have been introduced and studied by Cantero et al. [1].

Recall that a Schur function $s_+(z)$, $|s_+(z)| \leqslant 1$, $z \in \mathbb{D}$, (a finite Blaschke product is a special case) is in a one-to-one correspondence with the so-called Schur parameters

$$s_+(z) \sim \{a_0, a_1, \ldots\}, \tag{3}$$

where

$$s_+(z) = \frac{a_0 + z s_+^{(1)}(z)}{1 + z\overline{a_0} s_+^{(1)}(z)},$$

$a_0 = s_+(0)$, $a_1 = s_+^{(1)}(0)$, and so on...

A Schur class function has the representation

$$s_+(z) = \prod \frac{z_k - z}{1 - z\overline{z_k}} \frac{|z_k|}{z_k} e^{-\int \frac{t+z}{t-z} d\nu(t) + iC}, \tag{4}$$

where the set of zeros satisfies the Blaschke condition

$$\sum (1 - |z_k|^2) < \infty$$

and ν is a non-negative measure on $\mathbb{T}$. By $\sigma_{\text{ess}}(s_+)$ we denote the union of the limit points of $\{z_k\}$ with the support of ν. It is a closed subset of $\mathbb{T}$; if $\mathbb{T} \setminus \sigma_{\text{ess}}(s_+) \neq \emptyset$, the Schur function has an analytic extension through this open set to the exterior of the unit disk by the symmetry principle $s_+(z) = \overline{1/s_+(1/\bar{z})}$.

It is evident that the matrix $\mathfrak{A}(\{a_k\})$ is well defined by the two Schur functions $\{s_+(z), s_-(z)\}$, given by (3) and

$$s_-(z) \sim \{-\overline{a_{-1}}, -\overline{a_{-2}}, \ldots\}. \tag{5}$$

Basically from the identity, see (14) below,

$$\left\langle 0 \left| \frac{\mathfrak{A}+z}{\mathfrak{A}-z} \right| 0 \right\rangle = \frac{1+zs_+(z)s_-(z)}{1-zs_+(z)s_-(z)}$$

it follows

Theorem 1.1. *An arc* (a, b) *is free of the spectrum of the matrix* $\mathfrak{A}(\{a_k\})$ *if and only if* $(a, b) \subset \mathbb{T} \setminus \sigma_{\text{ess}}(zs_+(z)s_-(z))$ *and*

$$1 - zs_+(z)s_-(z) \neq 0 \tag{6}$$

on (a, b).

A CMV matrix $\mathfrak{A}$ is called almost periodic (a.p.) if the generated sequence $\{a_k\}_{k=-\infty}^{\infty}$ is almost periodic.

In this work for a fixed homogeneous set E we give a complete description of the class

$$\mathfrak{A}(E) = \{\mathfrak{A} \text{ is a.p.} : \sigma(\mathfrak{A}) = \sigma_{\text{a.c.}}(\mathfrak{A}) = E\}. \tag{7}$$

That is the class is formed by almost periodic CMV matrices with the spectrum on E and such that the absolutely continuous spectrum is also E (as a byproduct we prove that every matrix of the class has purely a.c. spectrum).

An essential ingredient of the result is the following counterpart of the Kotani–Pastur–Ishii theorem that was proved in [5].

Theorem 1.2. *Let* $s_\pm$ *be the Schur functions related to a CMV matrix* $\mathfrak{A}$. *Then*

$$\overline{zs_+(z)} = s_-(z) \tag{8}$$

a.e. on $\sigma_{a.c.}(\mathfrak{A})$.

Having in mind (6) and (8) let us introduce the following subclass of Schur functions.

Definition 1.3. Let E be a homogeneous set. We say that s_+ belongs to $S_+(E)$ if there exists a Schur function s_- such that (8) holds a.e. on E, $\sigma_{\text{ess}}(zs_+(z)s_-(z)) \subset E$ and (6) holds on $\mathbb{T} \setminus E$.

This set of functions one can parametrize by the following set of data. Let $\mathbb{T} \setminus E = \cup_{j \geqslant 0}(b_j^-, b_j^+)$. Let I_j be the double covering of the arc $[b_j^-, b_j^+]$

$$I_j := \{(t_j, \varepsilon_j) : t_j \in [b_j^-, b_j^+], \varepsilon_j = \pm 1\}$$

with identification $(b_j^\pm, 1) \equiv (b_j^\pm, -1)$. $D(E)$ is the direct product of I_j, that is $D \in D(E)$ is a formal sequence $D = \{(t_j, \varepsilon_j)\}_{j \geqslant 0}$.

For a given D we associate a function $M_D(z)$ with positive real part in two steps. First we define a function with a positive real part by its argument on $\mathbb{T}$:

$$W_D(z) = e^{i \int \frac{t+z}{t-z} f_D(t)\, dm(t)}, \tag{9}$$

where

$$f_D(t) = \begin{cases} -\pi/2, & t \in [b_j^-, t_j], \\ \pi/2, & t \in [t_j, b_j^+], \\ 0, & t \in E \end{cases}$$

and $m(t)$ is the Lebesgue measure on $\mathbb{T}$. Let us mention that, in fact, this function depends on $\{t_j\}$, not on $\{\varepsilon_j\}$. Having positive real part $W_D(z)$ admits the representation

$$W_D(z) = \int_E \frac{t+z}{t-z}\, d\sigma_D(t) + \sum_{\{j: t_j \neq b_j^{\pm}\}} \frac{t_j + z}{t_j - z} \sigma_{D,j} + i \Im W_D(0). \tag{10}$$

Define

$$M_D(z) = \frac{W_D(z) + \sum_{\{j: t_j \neq b_j^{\pm}\}} \frac{t_j+z}{t_j-z} \varepsilon_j \sigma_{D,j} - i \Im W_D(0)}{\Re W_D(0) + \sum_{\{j: t_j \neq b_j^{\pm}\}} \varepsilon_j \sigma_{D,j}}. \tag{11}$$

Theorem 1.4. *Given $s_+ \in S_+(E)$ there exists a unique $D \in D(E)$ such that*

$$\frac{1 + z s_+(z)}{1 - z s_+(z)} = M_D(z). \tag{12}$$

Conversely, for given $D \in D(E)$ (12) *defines a function of $S_+(E)$.*

Based on the theory developed in [22] we shall prove another representation for $S_+(E)$ functions in Theorem 4.1, and consequently the following main theorem:

Theorem 1.5. *Let $E \subset \mathbb{T}$ be a homogeneous set. Let Γ^* be the (compact Abelian) group of characters of the fundamental group of the domain $\overline{\mathbb{C}} \setminus E$. There exist a continuous map from $\Gamma^* \times \mathbb{T}$ to $\mathfrak{A}(E)$ such that*

$$\mathfrak{A}(E) = \{\mathfrak{A} = \mathfrak{A}(\alpha, \tau) : (\alpha, \tau) \in \Gamma^* \times \mathbb{T}\}.$$

Moreover, if $\mathfrak{A}(\alpha, \tau) = \mathfrak{A}_0(\alpha, \tau)\mathfrak{A}_1(\alpha, \tau)$, then

$$S^{-1}\mathfrak{A}_1(\alpha, \tau)\mathfrak{A}_0(\alpha, \tau) S = \mathfrak{A}(\mu\alpha, \xi\tau),$$

where (μ, ξ) is a fixed element of the group $\Gamma^* \times \mathbb{T}$. In other words a shift of the generated sequence of coefficients by one element has as a consequence the shift of the parametrizing element of the group $\Gamma^* \times \mathbb{T}$ by the element (μ, ξ).

The above theorem is formulated as a kind of “existence” theorem, however our result, in fact, is quite explicit. The parametrization is given in terms of special functions—the reproducing kernels of appropriate Hardy spaces on the domain $\overline{\mathbb{C}}\setminus E$. It is worth to mention that if E is a finite system of arcs these reproducing kernels have the standard representation in terms of Theta-functions [3].

For a discussion of the whole topic see the monograph by Simon [20,21]. In short, the study of Schur, respectively, Caratheodory functions which are rational on the Riemann surface $\mathcal{R}(E)$ and of the associated Verblunsky coefficients and orthogonal polynomials was initiated by Geronimus [8] and continued by Tomchuk [24] and in a series of papers by Peherstorfer and Steinbauer (see [16] and the references given therein). Concerning the Verblunsky coefficients it turned out that they are pseudo-periodic if and only if the harmonic measures of the arcs are rational. Geronimo and Johnson [6] considering the so-called two-sided case, proved then the almost periodicity of the Verblunsky coefficients. For Theta-function representation of the Verblunsky coefficients, of the minimum deviation etc. see [4,13,15,12]. We are thankful to a referee who draw our attention to the important papers [7,11]. It might be useful to look at the subject in the frame of the general theory of multi-diagonal matrices [14], see also [19]. The preprint version of this paper is [17].

2. Some spectral theory

2.1. Proof of Theorem 1.1

Let us mention the following representation for $s_+(z)$ in terms of its Schur parameters a_k's:

$$s_+(z) = \langle 0| \begin{bmatrix} \overline{a_0} & \rho_0 & & & & \\ \rho_0 & -a_0 - za_1 & -z\rho_1 & & & \\ & -z\rho_1 & \overline{a_2} + z\overline{a_1} & \rho_2 & & \\ & & \rho_2 & -a_2 - za_3 & -z\rho_3 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}^{-1} |0\rangle. \tag{13}$$

Let $\mathfrak{A}$ be the CMV matrix (2). Then, using (13), by a direct computation we get

$$\mathcal{E}^* \frac{\mathfrak{A}+z}{\mathfrak{A}-z}\mathcal{E} = \frac{I + zA_{-1}^* \begin{bmatrix} s_-^{(1)}(z) & 0 \\ 0 & s_+(z) \end{bmatrix}}{I - zA_{-1}^* \begin{bmatrix} s_-^{(1)}(z) & 0 \\ 0 & s_+(z) \end{bmatrix}}$$
$$= \begin{bmatrix} \dfrac{1 + zs_+^{(-1)}(z)s_-^{(1)}(z)}{1 - zs_+^{(-1)}(z)s_-^{(1)}(z)} & * \\ * & \dfrac{1+zs_+(z)s_-(z)}{1-zs_+(z)s_-(z)} \end{bmatrix}, \tag{14}$$

where $\mathcal{E} : \mathbb{C}^2 \to l^2(\mathbb{Z})$,

$$\mathcal{E}\begin{bmatrix} c_{-1} \\ c_0 \end{bmatrix} = |-1\rangle c_{-1} + |0\rangle c_0;$$

and the Schur functions $s_\pm(z)$ are defined by (3) and (5). Therefore if (a,b) is a spectral gap, the function $\frac{1+zs_+(z)s_-(z)}{1-zs_+(z)s_-(z)}$ has an analytic extension through this gap and (6) is also proved.

Conversely, since

$$s_+^{(-1)}(z) = \frac{a_{-1}+zs_+(z)}{1+z\overline{a_{-1}}s_+(z)}, \quad zs_-^{(1)}(z) = \frac{\overline{a_{-1}}+s_-(z)}{1+a_{-1}s_-(z)}, \tag{15}$$

we have

$$1-zs_+^{(-1)}(z)s_-^{(1)}(z) = \frac{\rho_{-1}^2(1-zs_+(z)s_-(z))}{(1+a_{-1}s_-(z))(1+z\overline{a_{-1}}s_+(z))}.$$

Therefore both and, in fact, all diagonal entries of the resolvent $\frac{\mathfrak{A}+z}{\mathfrak{A}-z}$ have analytic extension through the gap (a,b) and are real-valued there. That is (a,b) is a spectral gap and the theorem is proved. □

Let us note that due to (15) property (8) is shift invariant.

2.2. *Proof of Theorem 1.4*

Since $|E|>0$ the function s_+ determines s_- uniquely by (8). Define

$$W(z) = \frac{1}{2}\left\{\frac{1+zs_+(z)}{1-zs_+(z)} + \frac{1+s_-(z)}{1-s_-(z)}\right\} = \frac{1-zs_+(z)s_-(z)}{(1-zs_+(z))(1-s_-(z))}. \tag{16}$$

It is a function with a positive real part that, due to (8), is real valued a.e. on E and takes pure imaginary values on $\mathbb{T}\setminus E$, since $\sigma_{\text{ess}}(zs_+(z)s_-(z)) \subset E$. Moreover, due to condition (6) $W(z)$ has no zero in the spectral gap (b_j^-, b_j^+) for every j. It means that $\Im W(z)$ can switch its sign in the gap only once (recall that $\Im W(z)$ should decrease between each two poles here, since it is a Caratheodory function). We define $t_j \in [b_j^-, b_j^+]$ such that $\Im W(z) < 0$ in $[b_j^+, t_j]$ and $\Im W(z) > 0$ in $[t_j, b_j^+]$. One of these sets can be empty that is t_j can be the left or right endpoint of $[b_j^-, b_j^+]$. Thus the argument of W is defined completely (a.e. on $\mathbb{T}$) and (see (9)) we get that $W = CW_D$ with $C>0$.

Since both functions at the right-hand side of (16) have positive real part this representation of W is related to a certain representation of the positive measure in (10) as a sum of two positive measures.

The key claim is that for a homogeneous set E the measure σ_D on E is absolutely continuous for an arbitrary choice of t_j in $[b_j^-, b_j^+]$. A proof of this claim can be found in [18, Lemma 2.4]. Now, due to (8), the real parts, and therefore the density of the absolutely continuous components, are the same for both functions in (16) on E. Thus we have to decompose this part of the measure equally.

Let us discuss the distribution of the point mass $\sigma_{D,j}$ at t_j. This should be done only if $t_j \neq b_j^\pm$. It is evident that $t_js_+(t_j) = 1$ if a certain part of the mass was given to the first function and $s_-(t_j) = 1$ if part of $\sigma_{D,j}$ was given to the second function. However $1-zs_+(z)s_-(z)$ should not be zero on (b_j^-, b_j^+). Therefore the only possible way is to distribute the whole mass to the first or to the second function. We define

$$\varepsilon_j = \begin{cases} 1, & t_j s_+(t_j) = 1, \\ -1, & s_-(t_j) = 1. \end{cases}$$

Summarizing this mass distribution on E and at t_j's we obtain

$$\frac{1 + z s_+(z)}{1 - z s_+(z)} = A M_D(z) + i B$$

with a positive A, a real B and M_D given by (11). Due to the normalization in the origin we get $A = 1$, $B = 0$.

Thus (12) and the uniqueness is proved.

Now to the converse. A given D we associate M_D according to (11). This function has positive real part in $\mathbb{D}$ and is normalized by $M_D(0) = 1$. Therefore it is of form (12) with a certain Schur function $s_+(z)$. Now, consider $\overline{M_D(z)}$ for $z \in E$. Since $\overline{W_D(z)} = W_D(z)$ on E and

$$\overline{\frac{t_j + z}{t_j - z}} = \frac{z + t_j}{z - t_j}, \quad z \in \mathbb{T},$$

we get

$$\overline{M_D(z)} = \frac{W_D(z) - \sum_{\{j : t_j \neq b_j^{\pm}\}} \frac{t_j + z}{t_j - z} \varepsilon_j \sigma_{D,j} + i \Im W_D(0)}{\Re W_D(0) + \sum_{\{j : t_j \neq b_j^{\pm}\}} \varepsilon_j \sigma_{D,j}}.$$

The function in the RHS has an analytic continuation in $\mathbb{D}$, moreover, due to (10), it also has positive real part in $\mathbb{D}$. Therefore, there exists a Schur function s_- such that

$$\overline{M_D(z)} = \frac{1 + s_-(z)}{1 - s_-(z)}, \quad z \in E.$$

Thus $\overline{z s_+(z)} = s_-(z)$ on E.

Further, since M_D takes pure imaginary values at the gaps and is analytic there, except for a possible single pole in each gap, we have $\sigma_{\text{ess}}(s_+) \subset E$. The same holds true for s_- and, therefore, for the product $z s_+(z) s_-(z)$.

Finally, due to (16), we get that $1 - z s_+(z) s_-(z)$ has no zeros at the gaps (the cancelling of zeros with $1 - z s_+(z)$ and $1 - s_-(z)$ is impossible since the last two functions have no common zero in a gap). Thus condition (6) is also proved. □

3. The special function's representation

3.1. Character-automorphic Hardy spaces

Let E be a homogeneous set. The domain $\overline{\mathbb{C}} \setminus E$ is conformally equivalent to the quotient of the unit disk by the action of a discrete group $\Gamma = \Gamma(E)$. Let $z : \mathbb{D} \to \overline{\mathbb{C}} \setminus E$ be a covering map, $z \circ \gamma = z$, $\forall \gamma \in \Gamma$. In what follows we assume the following normalization to be hold

$$z : (-1, 1) \to (b_0^-, b_0^+) \subset \mathbb{T} \setminus E,$$

where $\mathbb{T} \setminus E = \cup_{j \geqslant 0}(b_j^-, b_j^+)$. In this case one can choose a fundamental domain $\mathfrak{F}$ and a system of generators $\{\gamma_j\}_{j \geqslant 1}$ of Γ such that they are symmetric with respect to complex conjugation:

$$\overline{\mathfrak{F}} = \mathfrak{F}, \quad \overline{\gamma_j} = \gamma_j^{-1}.$$

Denote by $\zeta_0 \in \mathfrak{F}$ the preimage of the origin, $z(\zeta_0) = 0$, then $z(\overline{\zeta_0}) = \infty$. Let $B(\zeta, \zeta_0)$ and $B(\zeta, \overline{\zeta_0})$ be the Green functions with $B(\overline{\zeta_0}, \zeta_0) > 0$ and $B(\zeta_0, \overline{\zeta_0}) > 0$. Then

$$z(\zeta) = e^{ic} \frac{B(\zeta, \zeta_0)}{B(\zeta, \overline{\zeta_0})}. \tag{17}$$

It is convenient to rotate (if necessary) the set E and to think that $c = 0$. Note that $B(\zeta, \zeta_0)$ is a character-automorphic function

$$B(\gamma(\zeta), \zeta_0) = \mu(\gamma) B(\zeta, \zeta_0), \quad \gamma \in \Gamma,$$

with a certain $\mu \in \Gamma^*$. By (17) and $z(\gamma(\zeta)) = z(\zeta)$

$$B(\gamma(\zeta), \overline{\zeta_0}) = \mu(\gamma) B(\zeta, \overline{\zeta_0}), \quad \gamma \in \Gamma.$$

Recall that the space $A_1^2(\alpha)$, $\alpha \in \Gamma^*$, is formed by functions of Smirnov class in $\mathbb{D}$ such that

$$f|[\gamma](\zeta) := \frac{f(\gamma(\zeta))}{\gamma_{21}\zeta + \gamma_{22}} = \alpha(\gamma) f(\zeta), \quad \gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}$$

and

$$\|f\|^2 := \int_{\mathbb{T}/\Gamma} |f(t)|^2 \, dm(t) < \infty.$$

We denote by $k^\alpha(\zeta, \zeta_0)$ the reproducing kernel of this space and put

$$K^\alpha(\zeta, \zeta_0) := \frac{k^\alpha(\zeta, \zeta_0)}{\|k\|} = \frac{k^\alpha(\zeta, \zeta_0)}{\sqrt{k^\alpha(\zeta_0, \zeta_0)}}.$$

Notice that in our case $f(\zeta) \in A_1^2(\alpha)$ implies $\overline{f(\bar{\zeta})} \in A_1^2(\alpha)$ and therefore

$$K^\alpha(\zeta_0, \zeta_0) = K^\alpha(\overline{\zeta_0}, \overline{\zeta_0}).$$

3.2. A recurrence relation for reproducing kernels

We start with

Theorem 3.1. *Both systems*

$$\{K^\alpha(\zeta, \zeta_0), B(\zeta, \zeta_0) K^{\alpha\mu^{-1}}(\zeta, \overline{\zeta_0})\}$$

and

$$\{K^{\alpha}(\zeta,\overline{\zeta_0}),\, B(\zeta,\overline{\zeta_0})K^{\alpha\mu^{-1}}(\zeta,\zeta_0)\}$$

form an orthonormal basis in the two-dimensional space spanned by $K^{\alpha}(\zeta,\zeta_0)$ *and* $K^{\alpha}(\zeta,\overline{\zeta_0})$. *Moreover*

$$\begin{aligned} K^{\alpha}(\zeta,\overline{\zeta_0}) &= a(\alpha)K^{\alpha}(\zeta,\zeta_0)+\rho(\alpha)B(\zeta,\zeta_0)K^{\alpha\mu^{-1}}(\zeta,\overline{\zeta_0}),\\ K^{\alpha}(\zeta,\zeta_0) &= \overline{a(\alpha)}K^{\alpha}(\zeta,\overline{\zeta_0})+\rho(\alpha)B(\zeta,\overline{\zeta_0})K^{\alpha\mu^{-1}}(\zeta,\zeta_0), \end{aligned} \tag{18}$$

where

$$a(\alpha)=a=\frac{K^{\alpha}(\zeta_0,\overline{\zeta_0})}{K^{\alpha}(\zeta_0,\zeta_0)},\qquad \rho(\alpha)=\rho=\sqrt{1-|a|^2}. \tag{19}$$

Proof. Let us prove the first relation in (18). It is evident that the vectors $K^{\alpha}(\zeta,\zeta_0)$ and $B(\zeta,\zeta_0)K^{\alpha\mu^{-1}}(\zeta,\overline{\zeta_0})$ are orthogonal, normalized and orthogonal to all functions f from $A^2_1(\alpha)$ such that $f(\zeta_0)=f(\overline{\zeta_0})=0$, that is to functions that form an orthogonal complement to the vectors $K^{\alpha}(\zeta,\zeta_0)$ and $K^{\alpha}(\zeta,\overline{\zeta_0})$. Thus

$$K^{\alpha}(\zeta,\overline{\zeta_0})=c_1K^{\alpha}(\zeta,\zeta_0)+c_2B(\zeta,\zeta_0)K^{\alpha\mu^{-1}}(\zeta,\overline{\zeta_0}).$$

Putting $\zeta=\zeta_0$ we get $c_1=a$. Due to orthogonality we have

$$1=|a|^2+|c_2|^2.$$

Now, put $\zeta=\overline{\zeta_0}$. Taking into account that $K^{\alpha}(\zeta_0,\overline{\zeta_0})=\overline{K^{\alpha}(\overline{\zeta_0},\zeta_0)}$ and that by normalization $B(\overline{\zeta_0},\zeta_0)>0$ we prove that c_2 being positive is equal to $\sqrt{1-|a|^2}$.

Note that simultaneously we proved that

$$\rho(\alpha)=B(\overline{\zeta_0},\zeta_0)\frac{K^{\alpha\mu^{-1}}(\overline{\zeta_0},\overline{\zeta_0})}{K^{\alpha}(\overline{\zeta_0},\overline{\zeta_0})}.\qquad \square$$

Corollary 3.2. *A recurrence relation for reproducing kernels generated by the shift of* Γ^* *on the character* μ^{-1} *is of the form*

$$\begin{aligned} &B(\zeta,\zeta_0)\left[\, K^{\alpha\mu^{-1}}(\zeta,\zeta_0),\ -K^{\alpha\mu^{-1}}(\zeta,\overline{\zeta_0})\,\right]\\ &\quad=\left[\, K^{\alpha}(\zeta,\zeta_0),\ -K^{\alpha}(\zeta,\overline{\zeta_0})\,\right]\frac{1}{\rho}\begin{bmatrix}1 & a\\ \bar a & 1\end{bmatrix}\begin{bmatrix}z & 0\\ 0 & 1\end{bmatrix}. \end{aligned} \tag{20}$$

Proof. We write, recall (17),

$$\begin{aligned} &B(\zeta,\zeta_0)\left[\, K^{\alpha\mu^{-1}}(\zeta,\zeta_0),\ -K^{\alpha\mu^{-1}}(\zeta,\overline{\zeta_0})\,\right]\\ &\quad=\left[\, B(\zeta,\overline{\zeta_0})K^{\alpha\mu^{-1}}(\zeta,\zeta_0),\ -B(\zeta,\zeta_0)K^{\alpha\mu^{-1}}(\zeta,\overline{\zeta_0})\,\right]\begin{bmatrix}z & 0\\ 0 & 1\end{bmatrix}. \end{aligned}$$

Then, use (18). $\square$

Corollary 3.3. *Let*

$$s^{\alpha}(z) := \frac{K^{\alpha}(\zeta, \overline{\zeta_0})}{K^{\alpha}(\zeta, \zeta_0)}. \tag{21}$$

Then the Schur parameters of the function $\tau s^{\alpha}(z)$, $\tau \in \mathbb{T}$, *are*

$$\{\tau a(\alpha\mu^{-n})\}_{n=0}^{\infty}.$$

Proof. Let us note that (20) implies

$$s^{\alpha}(z) = \frac{a(\alpha) + z s^{\alpha\mu^{-1}}(z)}{1 + \overline{a(\alpha)} z s^{\alpha\mu^{-1}}(z)}$$

and that $|a(\alpha)| < 1$. Then we iterate this relation. Also, multiplication by $\tau \in \mathbb{T}$ of a Schur class function evidently leads to multiplication by τ of all Schur parameters. □

3.3. Example (one-arc case)

In this case $\overline{\mathbb{C}} \setminus E \simeq \mathbb{D}$, Γ is trivial, and

$$z = \frac{B(\zeta, \zeta_0)}{B(\zeta, \overline{\zeta_0})} = \frac{\frac{\zeta - \zeta_0}{1 - \zeta\overline{\zeta_0}} \overline{\left(\frac{\overline{\zeta_0} - \zeta_0}{1 - \overline{\zeta_0}^2}\right)}}{\frac{\zeta - \overline{\zeta_0}}{1 - \zeta\zeta_0} \left(\frac{\overline{\zeta_0} - \zeta_0}{1 - \zeta_0^2}\right)} = -\frac{\zeta - \zeta_0}{\zeta - \overline{\zeta_0}} \frac{1 - \zeta\zeta_0}{1 - \zeta\overline{\zeta_0}} \frac{1 - \overline{\zeta_0}^2}{1 - \zeta_0^2}. \tag{22}$$

That is

$$b_0^+ = z(1) = -\frac{1 - \zeta_0}{1 + \zeta_0} \frac{1 + \overline{\zeta_0}}{1 - \overline{\zeta_0}},$$

and $b_0^- = \overline{b_0^+}$. We can put $\zeta_0 = ir$, $0 < r < 1$. Then

$$b_0^+ = \left(\frac{2r}{1 + r^2} + i\frac{1 - r^2}{1 + r^2}\right)^2 = e^{2i\theta},$$

where

$$\sin\theta = \frac{1 - r^2}{1 + r^2}, \quad \theta \in (0, \pi/2).$$

Further, for such z

$$s(z) = \frac{K(\zeta, \overline{\zeta_0})}{K(\zeta, \zeta_0)} = \frac{\frac{1}{1 - \zeta\zeta_0}}{\frac{1}{1 - \zeta\overline{\zeta_0}}} = \frac{1 - \zeta\overline{\zeta_0}}{1 - \zeta\zeta_0}.$$

Thus

$$a = s(0) = \frac{1 - |\zeta_0|^2}{1 - \zeta_0^2} = \frac{1 - r^2}{1 + r^2} = \sin\theta.$$

The Schur parameters of the function $s_\tau(z) = \tau s(z)$ are

$$s_\tau(z) \sim \{\tau \sin\theta, \tau \sin\theta, \tau \sin\theta \dots\}.$$

3.4. Lemma on the reproducing kernel

Let us map (the unit circle of) the z-plane onto (the upper half-plane of) the λ-plane in such a way that $b_0^- \mapsto 1$, $z(0) \mapsto \infty$, $b_0^+ \mapsto -1$. In this way ($\zeta \mapsto z \mapsto \lambda$) we get the function $\lambda = \lambda(\zeta)$ such that

$$z = \frac{B(0,\zeta_0)}{B(0,\overline{\zeta_0})} \frac{\lambda - \lambda_0}{\lambda - \overline{\lambda_0}}, \quad \lambda_0 := \lambda(\zeta_0). \tag{23}$$

Lemma 3.4. *Let* $k^\alpha(\zeta) = k^\alpha(\zeta, 0)$ *and let* $B(\zeta) = B(\zeta, 0)$ *be normalized such that* $(\lambda B)(0) > 0$. *Denote by* μ_0 *the character generated by* B, *i.e.*, $B \circ \gamma = \mu_0(\gamma) B$. *Then*

$$k^\alpha(\zeta,\zeta_0) = (\lambda B)(0) \frac{\overline{k^\alpha(\zeta_0)} \frac{k^{\alpha\mu_0}(\zeta)}{B(\zeta)k^{\alpha\mu_0}(0)} - \overline{\frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0)k^{\alpha\mu_0}(0)}} k^\alpha(\zeta)}{\lambda - \overline{\lambda_0}}. \tag{24}$$

Proof. We start with the evident orthogonal decomposition

$$A_1^2(\alpha\mu_0) = \{k^{\alpha\mu_0}\} \oplus B A_1^2(\alpha).$$

We use this decomposition to obtain

$$\lambda B f = (\lambda B)(0) f(0) \frac{k^{\alpha\mu_0}(\zeta)}{k^{\alpha\mu_0}(0)} + B\tilde{f}, \quad \tilde{f} \in A_1^2(\alpha).$$

Dividing by B and using the orthogonality of the summands, we get

$$P_+(\alpha)\lambda f = \tilde{f} = \lambda f - (\lambda B)(0) f(0) \frac{k^{\alpha\mu_0}(\zeta)}{B(\zeta)k^{\alpha\mu_0}(0)}, \tag{25}$$

where $P_+(\alpha)$ is the orthoprojector onto $A_1^2(\alpha)$.

Thus, on the one hand, for arbitrary $f \in A_1^2(\alpha)$

$$\langle(\lambda - \lambda_0) f, k^\alpha(\zeta,\zeta_0)\rangle = \{P_+(\alpha)(\lambda - \lambda_0) f\}(\zeta_0). \tag{26}$$

By virtue of (25) we have

$$\begin{aligned}\{P_+(\alpha)(\lambda - \lambda_0) f\}(\zeta_0) &= \lambda(\zeta_0) f(\zeta_0) - (B\lambda)(0) f(0) \frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0)k^{\alpha\mu_0}(0)} \\ -\lambda_0 f(\zeta_0) &= -(B\lambda)(0) \frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0)k^{\alpha\mu_0}(0)} \langle f, k^\alpha\rangle.\end{aligned} \tag{27}$$

On the other hand, since the function λ is real on $\mathbb{T}$,

$$\begin{aligned}\langle(\lambda - \lambda_0) f, k^\alpha(\zeta,\zeta_0)\rangle &= \langle f, (\lambda - \overline{\lambda_0}) k^\alpha(\zeta,\zeta_0)\rangle \\ &= \langle f, P_+(\alpha)(\lambda - \overline{\lambda_0}) k^\alpha(\zeta,\zeta_0)\rangle.\end{aligned} \tag{28}$$

Comparing (26) and (27) with (28), we get

$$P_+(\alpha)(\lambda-\overline{\lambda_0})k^\alpha(\zeta,\zeta_0) = -\overline{(B\lambda)(0)\frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0)k^{\alpha\mu_0}(0)}}k^\alpha(\zeta).$$

Using (25) again, we get

$$(\lambda-\overline{\lambda_0})k^\alpha(\zeta,\zeta_0) - (B\lambda)(0)k^\alpha(0,\zeta_0)\frac{k^{\alpha\mu_0}(\zeta)}{B(\zeta)k^{\alpha\mu_0}(0)}$$
$$= -\overline{(B\lambda)(0)\frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0)k^{\alpha\mu_0}(0)}}k^\alpha(\zeta).$$

Since $k^\alpha(0,\zeta_0) = \overline{k^\alpha(\zeta_0)}$, we have

$$(\lambda-\overline{\lambda_0})k^\alpha(\zeta,\zeta_0)$$
$$= (B\lambda)(0)\left\{\overline{k^\alpha(\zeta_0)}\frac{k^{\alpha\mu_0}(\zeta)}{B(\zeta)k^{\alpha\mu_0}(0)} - \overline{\frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0)k^{\alpha\mu_0}(0)}}k^\alpha(\zeta)\right\}.$$

The lemma is proved. □

Corollary 3.5. *In the above introduced notations*

$$\begin{aligned} zs^\alpha(z) &= \frac{B(0,\zeta_0)}{B(0,\overline{\zeta_0})}\frac{\lambda-\lambda_0}{\lambda-\overline{\lambda_0}}\frac{K(\zeta,\overline{\zeta_0})}{K(\zeta,\zeta_0)} \\ &= \frac{B(0,\zeta_0)}{B(0,\overline{\zeta_0})}\frac{k^\alpha(\zeta_0)\frac{k^{\alpha\mu_0}(\zeta)}{B(\zeta)} - \frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0)}k^\alpha(\zeta)}{\overline{k^\alpha(\zeta_0)}\frac{k^{\alpha\mu_0}(\zeta)}{B(\zeta)} - \overline{\frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0)}}k^\alpha(\zeta)} \\ &= \frac{B(0,\zeta_0)k^\alpha(\zeta_0,0)}{B(0,\overline{\zeta_0})k^\alpha(0,\zeta_0)}\frac{r(\lambda;\alpha)-r(\lambda_0;\alpha)}{r(\lambda;\alpha)-\overline{r(\lambda_0;\alpha)}}, \end{aligned} \tag{29}$$

where

$$r(\lambda;\alpha) := \frac{(\lambda B)(0)}{B(\zeta)}\frac{k^\alpha(0)}{k^{\alpha\mu_0}(0)}\frac{k^{\alpha\mu_0}(\zeta)}{k^\alpha(\zeta)}. \tag{30}$$

Let us point out that functions (30) are important in the spectral theory of Jacobi matrices [22]. Note that they are normalized by

$$r(\lambda;\alpha) = \lambda + \cdots, \quad \lambda\to\infty.$$

Corollary 3.6. *Let*

$$\tau(\alpha) = \left\{\frac{B(0,\zeta_0)k^\alpha(\zeta_0,0)}{B(0,\overline{\zeta_0})k^\alpha(0,\zeta_0)}\right\}^{-1}.$$

Then

$$\frac{1+z\tau(\alpha)s^\alpha(z)}{1-z\tau(\alpha)s^\alpha(z)} = \frac{r(\lambda;\alpha)-\Re r(\lambda_0;\alpha)}{i\Im r(\lambda_0;\alpha)}. \tag{31}$$

Proof. By (29)

$$\frac{1+z\tau(\alpha)s^{\alpha}(z)}{1-z\tau(\alpha)s^{\alpha}(z)}=\frac{1+\frac{r(\lambda;\alpha)-r(\lambda_0;\alpha)}{r(\lambda;\alpha)-\overline{r(\lambda_0;\alpha)}}}{1-\frac{r(\lambda;\alpha)-r(\lambda_0;\alpha)}{r(\lambda;\alpha)-\overline{r(\lambda_0;\alpha)}}}=\frac{r(\lambda;\alpha)-\Re r(\lambda_0;\alpha)}{i\Im r(\lambda_0;\alpha)}. \qquad \square$$

4. The results

4.1. Parametrizing $S_+(E)$ by $\Gamma^* \times \mathbb{T}$

Theorem 4.1. *Let $s_+ \in S_+(E)$. Then there exists a unique $(\alpha, \tau) \in \Gamma^* \times \mathbb{T}$ such that $s_+(z) = \tau s^{\alpha}(z)$, that is, the Schur parameters of a function from $S_+(E)$ are of the form $\{\tau a(\alpha\mu^{-n})\}_{n=0}^{\infty}$.*

Proof. We only have to show that $s_+(z) = \tau s^{\alpha}(z)$ with a certain (α, τ) and to use Corollary 3.3.

First, we choose θ such that $1 - zs_+(z)e^{i\theta} = 0$ at $z = z(0) \in (b_0^-, b_0^+)$. Since $s_+(z) \in \mathbb{T}$ when $z \in (b_0^-, b_0^+)$, there exists a unique $e^{i\theta}$ that satisfies this condition. It is obvious but important, that $s_+(z)e^{i\theta} \in S_+(E)$, that is, there exists a unique $D_\theta \in D(E)$ such that

$$\frac{1+zs_+(z)e^{i\theta}}{1-zs_+(z)e^{i\theta}} = M_{D_\theta}(z). \tag{32}$$

Let us denote by $\tilde{E} = [-1, 1] \setminus \cup_{j\geqslant 1}(\tilde{b}_j^-, \tilde{b}_j^+)$ the closed set that corresponds to E in the λ-plane (see (23)) and by $\tilde{D}_\theta$ the collection $\{(\lambda_j, \varepsilon_j)\}_{j\geqslant 1}$ that corresponds to $D_\theta \in D(E)$. As it was proved in [22], given $\tilde{D}_\theta$ there exists a unique $\alpha \in \Gamma^*$ such that $\tilde{D}_\theta$ is "the divisor of poles" of a function of form (30):

$$2r(\lambda;\alpha)=\lambda-q(\alpha)+\sqrt{\lambda^2-1}\prod_{j\geqslant 1}\frac{\sqrt{(\lambda-\tilde{b}_j^-)(\lambda-\tilde{b}_j^+)}}{\lambda-\lambda_j}+\sum_{\{j:\lambda_j\neq\tilde{b}_j^{\pm},\}}\frac{\varepsilon_j\tilde{\sigma}_j}{\lambda_j-\lambda}, \tag{33}$$

where $q(\alpha)$ is real and

$$\tilde{\sigma}_k = -\sqrt{(\lambda_k^2-1)(\lambda_k-\tilde{b}_k^-)(\lambda_k-\tilde{b}_k^+)}\prod_{j\geqslant 1, j\neq k}\frac{\sqrt{(\lambda_k-\tilde{b}_j^-)(\lambda_k-\tilde{b}_j^+)}}{\lambda_k-\lambda_j}.$$

Thus, after the renormalization and the change of variables, we get

$$\frac{r(\lambda(z);\alpha)-\Re r(\lambda_0;\alpha)}{i\Im r(\lambda_0;\alpha)} = M_{D_\theta}(z) \tag{34}$$

with the chosen α. Comparing (32), (34) with (31) we get

$$s_+(z) = \tau(\alpha)e^{-i\theta}s^{\alpha}(z).$$

The theorem is proved. $\square$

4.2. Proof of the Main Theorem

Let $\mathfrak{A}(\{a_k\}) \in \mathfrak{A}(E)$ and $s_\pm$ be the associated Schur functions. Then $s_+ \in S_+(E)$ by Theorems 1.1 and 1.2. By Theorem 4.1 there exists a unique $(\alpha, \tau) \in \Gamma^* \times \mathbb{T}$ such that

$$a_k = \tau a(\alpha\mu^{-k}), \tag{35}$$

where the function $a(\alpha)$ on Γ^* is given by (19). Note that the shifted sequence is of the form $\{\tau a(\alpha\mu\mu^{-k})\}_k$ that is $\mathfrak{A}(\{a_{k-1}\}) = \mathfrak{A}((\alpha, \tau) \cdot (\mu, 1))$.

Conversely, define the coefficient sequence by (35). Then by Corollaries 3.3, 3.6 $s_+ \in S_+(E)$ and $\sigma(\mathfrak{A}(\{a_k\})) = E$. As it was mentioned, due to Lemma 2.4 [18], the spectrum of $\mathfrak{A}(\{a_k\})$ is purely absolutely continuous. The key point is that the function $K^\alpha(\zeta_1, \zeta_2)$ is continuous on the compact Abelian group Γ^* ($\zeta_{1,2} \in \mathbb{D}$ are fixed) [9,22] and $K^\alpha(\zeta_0, \zeta_0) \neq 0$. Thus $a(\alpha)$ is a continuous function, and sequence (35) is almost periodic. Therefore $\mathfrak{A} \in \mathfrak{A}(E)$. □

4.3. Final remarks

(1) We would like to mention that a CMV matrix from $\mathfrak{A}(E)$ has a functional model (realization) as the multiplication operator in a certain space of character automorphic forms $L^2_1(\alpha)$ with the Hardy space $A^2_1(\alpha)$ related to $l^2(\mathbb{Z}_+)$. Moreover, this is basically the same realization that appears in [22] and [23], the spectral parameters are related by simple linear-fractional transforms, like (23). However, to get CMV, Jacobi matrices or a Sturm–Liouville operator we have to decompose this functional space in different basis systems (generally, chains of subspaces). Correspondingly the shift of the generated sequences (or potential in the Sturm–Liouville case) is related to the shift of the parameter in different directions on Γ^*. These directions are generated by character automorphic functions having different specific properties.

To be more precise let us compare the CMV and Jacobi cases. Recall that in [22] we have the system (for notations see Lemma 3.4)

$$\{B^n(\zeta)K^{\alpha\mu_0^{-n}}(\zeta)\}, \quad \text{where } K^\alpha(\zeta) = \frac{k^\alpha(\zeta)}{\|k^\alpha\|}, \tag{36}$$

as the basis in $A^2_1(\alpha)$ for $n \geqslant 0$ and in the whole $L^2_1(\alpha)$ for $n \in \mathbb{Z}$. The multiplication operator by $\lambda(\zeta)$ in $L^2_1(\alpha)$ with respect to this basis is the Jacobi matrix $J(\alpha)$. Then, as it easy to see, the shift $J(\alpha) \mapsto SJ(\alpha)S^{-1}$ corresponds to $\alpha \mapsto \mu_0\alpha$, where $\mu_0 \in \Gamma^*$ is generated by the character of the function $B(\zeta)$. The shift in the "CMV direction" μ, generally speaking, is different: it is generated by $B(\zeta, \zeta_0)$ (or, what is the same, $B(\zeta, \overline{\zeta_0})$). Let us describe briefly how "to shift" $J(\alpha)$ in the μ-direction.

First, we introduce $\Phi_1(\alpha)$ as the multiplication operator

$$B(\zeta, \zeta_0) : L^2_1(\alpha) \to L^2_1(\alpha\mu), \tag{37}$$

with respect to the basis system (36) related to each of the space. Respectively, $\Phi_2(\alpha)$ corresponds to $B(\zeta, \overline{\zeta_0}) : L^2_1(\alpha) \to L^2_1(\alpha\mu)$. It is important to note that both functions are analytic, and hence, the both operators are lower-triangular.

Then, according to (17) and (23), we have

$$\frac{J(\alpha) - \lambda_0}{J(\alpha) - \overline{\lambda_0}} = \Phi_2^{-1}(\alpha)\Phi_1(\alpha) \tag{38}$$

that is, (38) is an upper–lower-triangular factorization of the given unitary matrix. Finally, we claim that

$$J(\alpha\mu) = \Phi_1(\alpha)J(\alpha)\Phi_1^{-1}(\alpha). \tag{39}$$

Indeed, let $\hat{f} \in l^2(\mathbb{Z})$. Denote by $f \in L_1^2(\alpha\mu)$ the corresponding Fourier transform of $\hat{f}$ with respect to the basis of form (36). Then to apply $\Phi_1^{-1}(\alpha)$ to $\hat{f}$ means to multiply f by $B(\zeta, \zeta_0)^{-1}$ and then to decompose the result in the basis in $L_1^2(\alpha)$. Doing all three steps we come to the conclusion that we have to multiply f by $\lambda(\zeta)$ and to decompose the resulting function in the basis in $L_1^2(\alpha\mu)$. But this is exactly the way how $J(\alpha\mu)$ acts on $\hat{f}$.

Thus the summary is: to get $J(\alpha\mu)$ from $J(\alpha)$ we need to consider the function of $J(\alpha)$, see the LHS in (38); to decompose this matrix into an upper–lower-triangular factorization; to conjugate the initial $J(\alpha)$ using one of the triangle matrices, see (39). Of course, such a transformation looks strongly as a discrete version of a flow from the Toda-hierarchy. The remaining key problem to study factorization (38) we leave here as an open problem.

(2) In the main part of the paper we used the normalization $c = 0$ in (17). Generally it should not be zero, but it is evident that having (3) we get

$$s_+(e^{ic}z) \sim \{a_0, a_1 e^{ic}, a_2 e^{2ic}, \ldots\}.$$

Therefore, in the general case the parameter will be $\xi = e^{ic}$ in the main theorem. From this point of view our normalization corresponds to the case when the second parameter in $\Gamma^* \times \mathbb{T}$ is stable under the shift.

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